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Technical Note

Improved approximation for the Nusselt number for hydrodynamically developed laminar flow between parallel plates

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Abstract

Heat transfer for fully developed flow between parallel plates has been investigated widely throughout literature. Results are given in form of either piecewise or continuos functions for the local and overall Nusselt-number. While the first type of correlation might be more accurate the latter is easier to use. On the other hand there is no sense in correlating local and overall Nusselt-number separately, as they are connected by definition. Therefore we derived a correlation for the overall Nusselt-number to the above named problem from the series solution of the temperature field, which is continuos and more accurate than any other correlation known to us. The corresponding local Nusselt-number is easily calculated from that correlation. The result is, too, better than from any other correlation known to the authors. © 2002 Published by Elsevier Science Ltd.

1. Introduction

Heat transfer for fully developed flow between parallel plates has been investigated widely throughout the relevant literature. For the constant temperature boundary condition $\Theta_\mathrm{w} = 0$, correlations for the mean Nusselt number Nu, defined with the caloric mean temperature at the outlet $\bar{\Theta}_1$ by

$$\bar{\Theta}_1 = \exp(-\mathrm{NTU}) = \exp\left(-4\frac{Nu}{Gz}\right) \Longleftrightarrow Nu = -\frac{1}{4}Gz\ln(\bar{\Theta}_1) \tag{1}$$

and the local Nusselt number Nu_x defined by

$$Nu = \frac{1}{X}\int_0^X Nu_x\,\mathrm{d}x \Longleftrightarrow Nu_x = Nu - Gz\frac{\partial Nu}{\partial Gz} \tag{2}$$

may be found in [2,3] for example.

Shah and London [3] propose the piecewise defined function

$$Nu_\mathrm{SL} = \begin{cases} Nu_1 + 0.0235Gz & Gz < 166.\bar{6} \\ Nu_2 + 0.6 & 166.\bar{6} \geqslant Gz < 2000 \\ Nu_2 & Gz \geqslant 2000 \end{cases} \tag{3}$$

with

$$Nu_1 = \lim_{Gz\to 0}(Nu) = 7.541 \quad \text{and} \quad Nu_2 = \lim_{Gz\to\infty}(Nu) = 1.849Gz^{1/3} \tag{4}$$

for the mean Nusselt number and

$$Nu_{\mathrm{SL},x} = \begin{cases} Nu_1 + 6.874(Gz/1000)^{0.488}\exp\left(-\frac{245}{Gz}\right) \\ \quad Gz < 1000 \\ 2/3Nu_2 + 0.4 \\ \quad Gz \geqslant 1000 \end{cases} \tag{5}$$

for the local Nusselt number.

The VDI-Wärmeatlas [2] on the other hand gives the continuous correlation

$$Nu_\mathrm{WA} = (Nu_1^3 + Nu_2^3)^{1/3} \tag{6}$$

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Nomenclature

B	width of the plates (m)
b	constant (dimensionless)
c	heat capacity at constant pressure J/(kg K)
C_i	eigencoefficient (dimensionless)
d_h	hydraulic diameter $d_h = 2s$ (m)
Gz	Graetz number, $Gz = 2(\dot{M}cd_h)/(\lambda BL)$ (dimensionless)
L	length of the duct (m)
T	temperature (K)
$\dot{M}$	mass flow rate (kg/s)
n	exponent (dimensionless)
Nu	mean Nusselt number, $Nu = \alpha d_h/\lambda$ (dimensionless)
Nu_x	local Nusselt number (dimensionless)
NTU	number of transfer units (dimensionless)
s	distance of the plates, width of the channel (m)
u	velocity (m/s)
x	lateral coordinate, $x = -s/2, \ldots, s/2$ (m)
z	axial coordinate, $z = 0, \ldots, L$ (m)
Greek symbols	
α	heat transfer coefficient W/(m^2 K)
β_i	eigenvalue (dimensionless)
λ	heat conductivity W/(m K)
Θ	dimensionless temperature, $\Theta = (T - T_W)/(T_0 - T_W)$
ξ	dimensionless lateral coordinate, $\xi = 2x/s = -1, \ldots, 1$
ζ	dimensionless axial coordinate, $\zeta = z/L$
Subscripts	
0	at the inlet ($\zeta = 0$)
1	at the outlet ($\zeta = 1$)
W	at the wall

with

$$Nu_1 = \lim_{Gz\to 0}(Nu) = 7.541 \quad \text{and} \quad Nu_2 = \lim_{Gz\to\infty}(Nu) = 1.841Gz^{1/3} \tag{7}$$

for the mean Nusselt number, but has no separate correlation for the local Nusselt number available nor does it suggest the use of Eq. (2) to obtain it.

While Eq. (6) has the advantage of being continuous, Eq. (3) of Shah and London might be more accurate. Furthermore the numerical factor in Nu_2 is different in both correlations. As Nu_2 is an asymptote, there should only be one correct value. On the other hand Eq. (2) suggests that there is no need to correlate Nu and Nu_x separately as they are connected by definition.

Eq. (6) may be improved by introducing an additive constant b and choosing a different exponent n to give

$$Nu_{\text{new}} = (Nu_1^n + b^n + (Nu_2 - b)^n)^{1/n} \tag{8}$$

For laminar flow through a circular tube this type of correlation proved to be very accurate, with a maximum error lower than 1% for any Graetz number (see [2]). Thus, the correlation (8) would still fulfill the limiting cases of Gz tending to zero and to infinity, respectively, and might be more accurate for intermediate Graetz numbers. The corresponding local Nusselt number Nu_x follows by introducing (8) into (2).

2. Calculation from the series solution

To validate the assumption and to fit the constant b and the exponent n we have to calculate the Nusselt number from the infinite series solution for the temperature profile in fully developed laminar flow through a flat duct with constant temperature boundary condition

$$\Theta = \sum_{i=1}^{\infty} C_i M\left(\frac{1}{4} - \frac{\beta_i}{4}, \frac{1}{2}, \beta_i\xi^2\right) \exp\left(-\frac{\beta_i}{2}\xi^2 - \frac{32\beta_i^2}{3Gz}\zeta\right) \tag{9}$$

wherein $M(a, b, c)$ is the Kummer function [1]. The eigenvalues β_i may be calculated from

$$\Theta(\zeta, \xi = 1) = 0 = \sum_{i=1}^{\infty} C_i M\left(\frac{1}{4} - \frac{\beta_i}{4}, \frac{1}{2}, \beta_i\right) \exp\left(-\frac{\beta_i}{2} - \frac{32\beta_i^2}{3Gz}\zeta\right) \tag{10}$$

and the corresponding coefficients follow from

$$C_i = \int_0^1 \left(G_i(1 - \xi^2)\right) \mathrm{d}\xi \Big/ \int_0^1 \left(G_i^2(1 - \xi^2)\right) \mathrm{d}\xi = \bar{G}_i/\bar{G}_{2,i} \tag{11}$$

with

$$G_i = \exp\left(-\frac{1}{2}\beta_i\xi^2\right) M\left(\frac{1}{4} - \frac{\beta_i}{4}, \frac{1}{2}, \beta_i\xi^2\right) \tag{12}$$

for constant temperature $\Theta_0 = 1$ at the inlet at $\zeta = 0$. Eigenvalues can be calculated with *Maple V Release 5* for instant. We did so, calculating up to 200 eigenvalues in less than an hour on a PIII 450 MHz processor. As the evaluation of the finite integrals in (11) needs much more computing power than finding solutions of (10), we could only calculate the first 96 corresponding eigencoefficients.

Because of the parabolic velocity distribution the caloric mean temperature

$$\bar{\Theta} = \int_0^1 u\Theta \,\mathrm{d}\xi \Big/ \int_0^1 u \,\mathrm{d}\xi \tag{13}$$

at the outlet ($\zeta = 1$) is then given by

$$\bar{\Theta}_1 = \bar{\Theta}(\zeta = 1) = \frac{3}{2}\sum_{i=1}^{\infty} C_i F_i \bar{G}_i \tag{14}$$

with

$$F_i = \exp\left(-\frac{32\beta_i^2}{3Gz}\right) \tag{15}$$

The Nusselt number follows to be

$$Nu = -\frac{1}{4} Gz \cdot \ln\left(\frac{3}{2}\sum_{i=1}^{\infty} C_i F_i \bar{G}_i\right) \tag{16}$$

and for Gz approaching zero Nu_1 is

$$\begin{aligned} Nu_1 &= \lim_{Gz\to 0}(Nu) = -\frac{1}{4} Gz \cdot \ln\left(\frac{3}{2} C_1 F_1 \bar{G}_1\right) = \frac{8}{3}\beta_1^2 \\ &= 7.541 \end{aligned} \tag{17}$$

because only the first term of the series solution (10) is significant in this limit. The limiting function Nu_2 for Gz approaching infinity cannot be calculated from the series solution, as the series converges very slowly for higher Graetz numbers. In this region, the asymptote Nu_2 can be calculated from Lévêque's solution to be

$$\begin{aligned} Nu_2 &= \frac{3}{2}\frac{2}{\Gamma(4/3)6^{1/3}} Gz^{1/3} = 1.84882587 Gz^{1/3}\ldots \\ &\approx 1.849 Gz^{1/3} \end{aligned} \tag{18}$$

(see [4, Chapter 3]). Therefore Nu_2 given in (4) by Shah and London [3] is more accurate than Nu_2 given in the VDI-Wärmeatlas [2].

The local Nusselt number follows from introducing (16) into (2) and evaluates to

$$Nu_x = Nu - Gz\frac{\partial Nu}{\partial Gz} = \frac{8}{3}\frac{\sum_{i=1}^{\infty} C_i \bar{G}_i F_i \beta_i^2}{\sum_{i=1}^{\infty} C_i \bar{G}_i F_i} \tag{19}$$

with the limiting cases

$$\begin{aligned} Nu_{1,x} &= \lim_{Gz\to 0}(Nu_x) = \frac{8}{3}\beta_1^2 = Nu_1 \quad \text{and} \\ Nu_{2,x} &= \lim_{Gz\to\infty}(Nu_x) = \frac{2}{3} Nu_2 \end{aligned} \tag{20}$$

3. Results

With 96 eigenvalues and eigencoefficients computed we found that our mean and local Nusselt numbers matched perfectly with those tabulated in [3] for Gz up to 10^5. For $Gz > 10^5$ simply more series terms are needed. By minimization of the maximal absolute relative error between those Nusselt numbers and the suggested correlation, we found

$$Nu_{\text{new correlation}} = \left(Nu_1^n + b^n + (Nu_2 - b)^n\right)^{1/n} \tag{21}$$

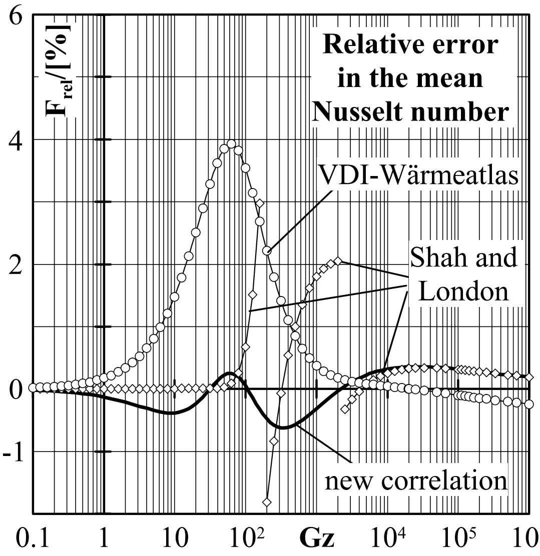

Fig. 1. Relative error in mean Nusselt number prediction.

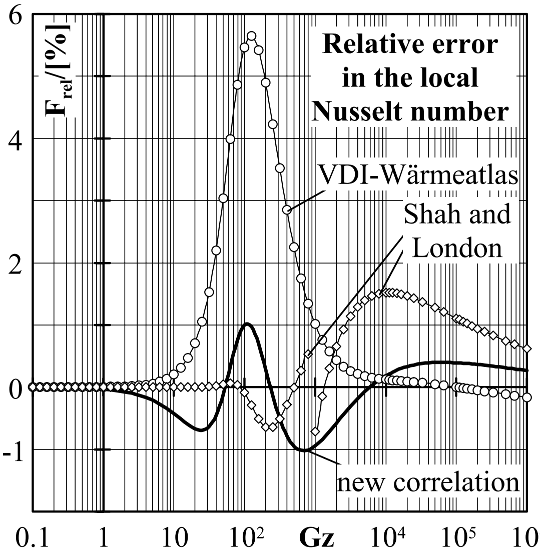

Fig. 2. Relative error in local Nusselt number prediction.

with

$$\begin{aligned} Nu_1 &= 7.541, \quad n = 3.592 \\ Nu_2 &= 1.841 Gz^{1/3}, \quad b = 0 \end{aligned} \tag{22}$$

which has a relative error as shown in Fig. 1 of less than 0.63% for any Graetz number. The corresponding local Nusselt number then follows from

$$Nu_x = Nu - \frac{1}{3}\left(\frac{Nu_2 - b}{Nu}\right)^{n-1} Nu_2 \tag{23}$$

and has a relative error as shown in Fig. 2 of less than 1.02% for any Graetz number. It should be noted that the value $b = 0$ was found by optimization of (b, n) for the plane duct, while $(b = 0.7, n = 3)$, or similar combinations $(b = 0.8, n = 2.8)$ have been found for the circular duct to be the optimal parameters in Eq. (21).

References

[1] M. Abramowitz, I.A. Stegun, Handbook of Mathematical Functions, fifth ed., Dover, New York, 1969.

[2] VDI-Wärmeatlas (8. Auflage), Springer, Berlin, 1997.

[3] R.K. Shah, A.L. London, Advances in heat transfer, in: T.F. Irvine Jr., J.P. Hartnett (Eds.), Advances in Heat Transfer (Suppl. 1), Academic Press, New York, 1978.

[4] S. Kakaç, R.K. Shah, W. Aung, Handbook of Single-Phase Convective Heat Transfer, Wiley, New York, 1987.